RABBI BEN EZRA ON PERMUTATIONS AND COMBINATIONS

By JEKUTHIEL GINSBURG.

INTRODUCTORY NOTE: There are few algebraic topics more interesting to high school pupils that Permutations and Combinations when this subject is presented in an elementary fashion. It has of late years dropped out of the curriculum because it had come to be unnecessarily difficult through numerous unnecessary complications, but as a slight diversion it still has value, and in advanced algebra it is almost necessary.

Some interesting material relating to its presentation in early times has recently been found by Messrs. Ginsburg and Turetsky, and the editor of THE MATHEMATICS TEACHER has asked me to write a brief statement concerning it. In studying some unpublished manuscripts of Rabbi Ben Ezra (the learned Hebrew scholar of the 12th century, who is the subject of one of Browning's poems), Mr. Ginsburg found a curious motive leading to the study of combinations, namely, the desire of the astrologers to find the number of ways in which the planets could come into conjunction, this having an important bearing upon astrological predictions. The treatment is entirely distinct from any now in use, and it has been set forth in print only in the Hebrew language, and very imperfectly. Mr. Ginsburg has compared the text with an unpublished manuscript in the Hebrew Theological Seminary of New York, and has shown some very interesting results to which the work may easily lead. His translation and commentary are set forth in this article, and the material found by Mr. Turetsky will appear in a subsequent issue.

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The early history of the theory of permutations and combinations is one of the least explored fields of scientific research. The information on the subject given in the standard works on the history of mathematics is hardly sufficient to give even an approximate idea of the gradual development of this beautiful doctrine. Our knowledge about it is limited to a number of isolated facts chiefly relating to Christian Europe in the late Middle Ages and in modern times. Until recently the period immediately preceding the awakening of the interest in science in Europe has been a *tabula rasa* so far as this branch of mathematics is concerned. What did the Arabic scholars and those who came under their influence know about permutations and combinations? What utilitarian or ideal needs first attracted the attention of scholars to this subject? What influence, if any, did their work exert on later European developments? The information at our disposal does not allow us to answer any of these questions with the slightest degree of certainty. Most of

them will have to remain unanswered until further progress is made in the study of the rich literary treasures in the large European collections of Arabic, Hebrew, Persian and Hindu manuscript works of that period.

The object of the present article is to establish a few new points of departure for the further study of the subject, by calling attention to a number of hitherto unannounced facts the records of which have been preserved in the Hebrew literature, and especially to an extremely interesting mathematical fragment in a manuscript of the influential twelfth century astrologer and mathematician Rabbi Abraham ben Ezra (1093-1167), or, as he is known to the English-speaking world, Rabbi Ben Ezra.[1]

This fragment, which deals with the theory of permutations, is not found, as might have been anticipated, in one of his mathematical treatises, but in a work on astrology in which he discusses the influence of the stars on the destinies of the world.[2] This is interesting because it suggests that Rabbi Ben Ezra did not see any use for the theory of permutations outside of astrology. Most probably he was not aware, while solving a practical question of astrology by what we now call the theory of permutations, that he was writing upon mathematics and that his speculations had any theoretical value at all. This leads us directly to, and reveals in operation, one of the first causes that brought about the interest in combinations—the all-powerful influence of astrology. This should not surprise us. The apparent multiplicity of the powers of nature as represented by the stars acting severally and in conjunction with each other naturally led the astrologers to consider various combinations of stellar bodies and their possible influence upon human life. A similar mystic belief in the powers of the letters of the Holy

[1] The form used by Browning in his famous poem.

[2] The name of the treatise is ha-Olam (the World). It is extant in manuscript form in a number of European libraries and also in the Jewish Theological Seminary in New York. In the New York copy the mathematical passage has been garbled by a somewhat ignorant scribe. Fortunately a portion of the work containing the passage referred to has been published in Hebrew from a manuscript in Berlin by D. Herzog in his edition (Tsophnath Pa-neakh) of Bonfil's supercommentary on Ben Ezra's work on the Bible (Heidelberg, 1911), with the view to elucidating an obscure passage in the supercommentary, but without pointing out its significance for the history of mathematics. In the following translation and discussion Herzog's printed version was used in conjunction with the New York manuscript copy.

Writ caused Jewish and Christian cabalists to develop a theory of permutations of their own, the details of which still await investigation. The work on permutations, written by one such cabalist, Moses Cordovero, has been the subject of careful study by Mr. Turetsky and will appear in a subsequent number of THE MATHEMATICS TEACHER.

The mathematical passage referred to is written in the vigorous and incisive style that characterizes the best of Ben Ezra's writings. The method of permutation used by him is original, and, as it seems to the writer of the present article, entirely unique.

Ben Ezra opens the discussion with a vigorous attack on the famous Arab astrologer Abu-Maschar,[1] who died in 886 A. D., about 100 years of age. "If thou hast found a book on conjunctions written by Abu-Maschar thou must not agree with him and thou shouldst not listen to him.[2] . . . Neither shalt thou trust, in the matter of conjunctions, to tables made by Hindu scholars, because they are altogether incorrect . . .[3] The right thing to do is to rely in each period upon tables made by contemporary scholars."

Ben Ezra then proceeds to discuss the number of possible conjunctions[4] between various members of the planetary system—a procedure that could not be very well undertaken without some scheme of combinations and permutations, and it is here that he develops his curious method.

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[1] Jafau ibn Mohammed ibn Omar al-Balkhi, Abu-Marschar. In the Latin Middle Ages he was known as Albumasar.

[2] His astronomical objections have been omitted in the translation as not bearing directly upon the subject of discussion.

[3] This testimony as to the state of Hindu science strikingly confirms the opinion held by a number of modern scholars who had less access to Hindu science than Ben Ezra. It also shows that the Hindu influence, which was very pronounced at the beginning of the Arabic period, faded out completely at the time of Ben Ezra.

[4] The astrologers believed the destinies of countries, nations, and individuals to be indicated in the heaven by the various positions of the planets. Two planets meeting at the same place in the heaven form a conjunction and exert a special significance on the development of events, the significance varying with the individual stars. A conjunction of three planets—that is, when three planets meet at the same place in heaven—has a greater influence than a conjunction of two planets, and so on. The most dreaded conjunction is of course the one of all seven planets referred to in an early Hindu tradition and expected to return in 26,000 years. A meeting of five planets was expected to cause great disturbances, inundations, plagues, etc., and the end of the world. For the year 1514 Stoffler predicted a terrible inundation on account of the conjunction of the superior planets. (See Hutton's Mathematical Dictionary.)

In view of the fact that this work by Ben Ezra has been published only in Hebrew and that no examination of its mathematical value has been made in any language, a complete translation and analysis is given below. In view of the peculiar style of Ben Ezra the explanations will be given after each closed paragraph.

"The number of conjunctions [of the seven planets] is 120. And this is how the number is found: It is known that the sum of the numbers from 1 to any desired number is found by multiplying it by half of itself and by $\frac{1}{2}$ of unity. For example if it is desired to find the number containing [all numbers] from 1 to 20, we multiply 20 by $\frac{1}{2}$ of itself which is ten [obtaining 200] and by $\frac{1}{2}$ of unity [*i. e.*, $20 \times \frac{1}{2} = 10$], and behold the result obtained is 210.[1]

"Now we shall proceed to find the number of binary conjunctions—that is, the combinations of two stars each. And it is known that there are seven planets. Now Jupiter has six conjunctions with the planets. Let us multiply then 6 by its half and by half of unity [$6 \times 3 = 18$, and $6 \times \frac{1}{2} = 3$]. The result is 21, and this is the number of the binary conjunctions."

It is characteristic of Ben Ezra's enigmatic way of writing that he should specify only the six conjunctions of Jupiter with the other planets, leaving for the reader to find out the source of the other members of the series

$$1 + 2 + 3 + 4 + 5 + 6 = 21$$

which he uses.[2]

"We wish now to know how many ternary combinations are possible. We begin by putting Saturn with Jupiter, and with them one of the others. The number of the others is five: mul-

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[1] In modern notation $1 + 2 + 3 + \ldots + n = n \cdot \frac{1}{2}n + n \cdot \frac{1}{2} = \frac{1}{2}n(n + 1)$.

[2] His reasoning was apparently this: Jupiter combines with each of the other six planets to form six conjunctions. Eliminating Jupiter the combinations of the remaining six planets with each other are now considered. Picking out one of them, for example Saturn, we find that, combining it with the other five planets, we obtain five combinations containing Saturn, and so on until the whole series $6 + 5 + 4 + 3 + 2 + 1$ is obtained.

tiply 5 by its half and by half of unity. The result is 15. And these are the conjunctions of Jupiter.[1]

"In the case of the conjunctions of Saturn[2] we have four planets left.[3] Multiply 4 by half of itself and by one-half. The result is 10.

"The conjunctions of Mars[4] are 3 multiplied by 2, amounting to 6.[5]

"The conjunctions of the sum[6] are 2 multiplied by $1\frac{1}{2}$ [that is by $\frac{1}{2}$ of 2 and by $\frac{1}{2}$], the result being 3,[7] and the con-

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[1] That is, Ben Ezra considered first the question as to how many of the ternary conjunctions contain Jupiter as a member. It is evidently the number of the conjunctions of the six other planets two at a time, since to each of these we may add Jupiter to form a ternary conjunction. But the number of these combinations $({}_6C_2)$ is, according to the previous reasoning, $5 \cdot \frac{1}{2} \cdot 5 + 5 \cdot \frac{1}{2} = 5\left(\frac{5}{2} + \frac{1}{2}\right) = \frac{6 \cdot 5}{2} = 15$. The fact that Ben Ezra mentioned Jupiter and Saturn at the beginning suggests that his reasoning in proving the truth of this proposition was as follows: Taking first Jupiter and Saturn, we have five planets left (including, as was the custom, the sun and the moon—Uranus and Neptune being then unknown), which by combining successively with the combination Jupiter-Saturn will give five combinations. Taking now Jupiter and Mars, for example, leaving out Saturn, we have four planets left, which will, in combination with Jupiter and Mars, give four new ternary conjunctions. Taking now Jupiter and the sun, leaving out Mars and Saturn, the three remaining planets will give us three new combinations. Hence the number of ternary conjunctions, each of which contains Jupiter, is $= 5 + 4 + 3 + 2 + 1 = 5 \cdot \frac{1}{2} \cdot 5 + 5 \cdot \frac{1}{2} = 15$.

[2] That is, ternary conjunctions containing Saturn but not Jupiter.

[3] Using the reasoning developed in the case of conjunctions of Jupiter: After all the conjunctions containing Jupiter have been accounted for we have only six planets left. To find the number of combinations containing Saturn, we first fix Saturn and one of the other planets, e. g., Mars. By joining each one of the four remaining planets to the pair Jupiter-Mars, we get four conjunctions. Now joining Saturn with Venus, and leaving out Mars, we get three more combinations. Hence the number of conjunctions containing Saturn and not Jupiter is $1 + 2 + 3 + 4 = 4 \cdot \frac{1}{2} \cdot 4 + 4 \cdot \frac{1}{2} = 2\frac{1}{2} \cdot 4 = 10$.

[4] That is, $1 + 2 + 3 = 3 \cdot \frac{4}{2} = 3 \cdot 2 = 6$, the reasoning being similar to the above.

[5] That is, conjunctions containing Mars but not Jupiter and Saturn.

[6] That is, containing the sun but not any of the previously mentioned planets.

[7] That is, $1 + 2 = \frac{(1 + 2)2}{2} = 3$. It is interesting that even in a simple case like this Ben Ezra insisted on considering the work as involving a series.

junctions of the [three] lower planets[1] are one. All together [the number] is 35, and this is the number of the ternary combinations."

Writing down the series of numbers obtained by Ben Ezra in such a way as to record the arithmetical operations used, we quite unexpectedly obtain the following result:

$$ {}_7C_3 = \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \frac{3\cdot 4}{2} + \frac{4\cdot 5}{2} + \frac{5\cdot 6}{2} $$

whence we get the beautiful relationship

$$ 2\cdot{}_1C_3 = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6, $$

a relationship that could be easily extended to ${}_8C_2$, ${}_9C_2$, etc. In general,

$$ 2\cdot{}_mC_3 = 1\cdot 2 + 2\cdot 3 + \ldots + (m-2)\,(m-1), $$

a relationship not very helpful for permutations but which supplies an elegant method for the summation of the series

$$ 1\cdot 2 + 2\cdot 3 + \ldots + (m-1)m. $$

In the computation of the quaternary conjunctions Ben Ezra follows the same line of reasoning:

"We wish now to obtain the number of quaternary conjunctions. We shall begin with Jupiter, Saturn, and Mars. And since it is necessary to have three planets to join with it, the conjunctions begin with four.[2] Multiply by $2\frac{1}{2}$ [*i. e.*, multiply 4 by $\frac{1}{2}$ of 4 and by $\frac{1}{2}$] and we obtain 10. Then follow the conjunction of Jupiter and Saturn with the others, and they will be multiplied by 2, amounting to 6; and behold it is now 16. Then Jupiter with Mars will begin, and there will be two [free planets] multiplied by $1\frac{1}{2}$. Three is obtained. Then one more conjunction, hence the number of conjunctions of Jupiter is 20.[3]

Saturn begins with three.[4]

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[1] That is, conjunctions not containing the other four.

[2] That is, at first we get conjunctions by adding each of the four remaining planets to the group Jupiter-Saturn-Mars. Eliminating one of the three, say Mars, and putting in its place one of the four left, we shall have three planets left giving rise to three new conjunctions.

[3] The details of these computations are similar to those in the case of ternary conjunctions and can easily be followed by using the reasoning given in the footnotes to that case. The number of quaternary combinations containing Jupiter must evidently be equal to the number of ternary conjunctions of the remaining six planets, that is ${}_6C_3 = \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 20$, which agrees with the result obtained by Ben Ezra.

[4] That is, excluding Jupiter, fixing three planets, say Saturn, Mars, and Venus, we have three possibilities of filling the fourth vacancy. Discarding one of the first three and replacing it by one of the unfixed three we get two choices, and so on, so that, in all, we have $3 + 2 + 1 = 6$.

Three times two is six.[1] Then 2 multiplied 1½ gives 3. Then one more conjunction. And then there are 10 conjunctions of Saturn.[2]

The number of conjunctions containing Mars but not Jupiter and Saturn is computed by Ben Ezra in the same way. "Mars begins with two non-fixed planets. Two times one and a half are three. Then one more conjunction. Together there are four quaternary conjunctions. Conjunction of the sun with the lower planets is one. All together, there are 35 quaternary conjunctions."

Ben Ezra's work, in computing the quaternary conjunctions, could be expressed by

$${}_7C_n = {}_6C_3 + {}_5C_3 + {}_4C_3 + {}_3C_3 = 20 + 10 + 4 + 1$$

An interesting question arising out of this discussion is this: Was Rabbi Ben Ezra aware of the fact that the numbers 20, 10, 4, 1 are respectively the values of the members of the series

$${}_7C_4 = \frac{1\cdot 2\cdot 3}{6} + \frac{2\cdot 3\cdot 4}{6} + \frac{3\cdot 4\cdot 5}{6} + \frac{4\cdot 5\cdot 6}{6}$$

which is analogous to the two series

$${}_7C_2 = 1 + 2 + 3 + 4 + 5 + 6,$$

$${}_7C_3 = \frac{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + 5\cdot 6}{1\cdot 2}$$

which he undoubtedly recognized as such? At the present stage of our knowledge we can only speculate upon the answer, although the brevity with which he treats the following cases suggests that he expected the reader to derive some sort of a rule from the previous discussion.

"We wish to find the quinary conjunctions. We find for Jupiter 15, for Saturn 5 and for Mars 1, together 21."[3]

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[1] $1 + 2 + 3 = 3 \cdot — = 3 \cdot 2 = 6.$

[2] That is, not counting Jupiter. The same number could be obtained by noting that the number of quaternary conjunctions containing Saturn, but not Jupiter, is ${}_5C_3 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10.$

[3] In modern terms: the number of conjunctions of planets containing Jupiter as a member is ${}_6C_4$, or 15. Those containing Saturn but not Jupiter are ${}_5C_4$, or 5. Those containing Mars but not the previous two are ${}_4C_4$, or 1.

Here again the numbers given by Ben Ezra fit into the identity

$$_7C_5 = \frac{1\cdot2\cdot3\cdot4}{4!} + \frac{2\cdot3\cdot4\cdot5}{4!} + \frac{3\cdot4\cdot5\cdot6}{4!},$$

which leads to the relationship

$$4!\,{}_7C_5 = 1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + 3\cdot4\cdot5\cdot6.$$

The remaining groups of conjunctions are treated not quite as fully as the previous ones. "The senary conjunctions are six for Jupiter and one for Saturn,[1] and there is one conjunction of all the seven planets."

This could be expressed in modern notation as follows:

$$_7C_6 = \frac{1\cdot2\cdot3\cdot4\cdot5}{5!} + \frac{2\cdot3\cdot4\cdot5\cdot6}{5!},$$

$$_7C_7 = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6}{6!}.$$

Rabbi Ben Ezra concludes with the following remark: "The total number is 120 conjunctions; all component [groups of] conjunctions are odd in number and they are divisible by 7."

A generalization of the above results leads to the following interesting relations:

$$_mC_2 = 1 + 2 + 3 + \ldots + (m-1) \qquad (1)$$

$$2!\,{}_mC_3 = 1\cdot2 + 2\cdot3 + 3\cdot4 + \ldots + (m-2)(m-1) \qquad (2)$$

$$3!\,{}_mC_4 = 1\cdot2\cdot3 + 2\cdot3\cdot4 + \ldots + (m-3)(m-2)(m-1) \qquad (3)$$

$$4!\,{}_mC_5 = 1\cdot2\cdot3\cdot4 + 2\cdot3\cdot4\cdot5 + \ldots + (m-4)(m-3)(m-2)(m-1) \qquad (4)$$

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$$r!\,{}_mC_{n+1} = 1\cdot2\cdot3\ldots r + 2\cdot3\ldots(r+1) + \ldots + (m-r+1)(m-r+2)\ldots(m-1)\ldots \qquad (5)$$

The first two were clearly recognized as series by Ben Ezra, and were treated by him as such; the others are suggested by the method he employed. Crude as these relations are for the subject itself, they are very helpful in the summation of series of products of natural numbers. In fact there is hardly another method as convenient and as easy to remember as the one ob-

[1] That is, counting Saturn but not Jupiter.

tained by substituting $m+1$ for m in (1), $m+2$ for m in (2), $m+3$ for m in (3), and so on. In this way we obtain the following set of formulas:

$$1+2+3+\ldots+m = {}_{m+1}C_2$$
$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+m(m+1) = 2!\,{}_{m+2}C_3$$
$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+m(m+1)(m+2) = 3!\,{}_{m+3}C_4$$

From this we can easily deduce the following general theorem: *The sum of the first m terms of the series of products of natural numbers taken in order of magnitude r at a time is equal to* $r!\,{}_{m+r}C_{r+1}$.

The above set of formulas allows us also to find the sum of the powers of the series of natural numbers in a more elegant way than is ordinarily used. We notice that (5) may be written

$$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+m(m+1) = 1(1+1)+2(2+1)+3(3+1)+m(m+1)$$
$$= (1^2+2^2+3^2+\ldots+m^2)+(1+2+3+\ldots m)$$
$$= 2!\,{}_{m+2}C_3.$$

Hence $1^2+2^2+3^2+\ldots+m^2 = 2!\,{}_{m+2}C_3 - (1+2+3+\ldots+m)$
$$= 2!\,{}_{m+2}C_3 - {}_{m+1}C_2.$$

For the sum of the cubes consider the identity (6)

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\ldots+m(m+1)(m+2) = 3!\,{}_{m+3}C_4$$

The general term being of the form $a(a+1)(a+2) = a^3+3a^2+a$, we may write it

$$(1^3+2^3+\ldots+m^3)+3(1^2+2^2+\ldots+m^2)+2(1+2+\ldots+m) = 3!\,{}_{m+3}C_4.$$

Substituting for the sum of the squares and first powers the values previously obtained, and then transposing, we have

$$1^3+2^3+\ldots+m^3 = 3!\,{}_{m+3}C_4 - 3\cdot 2!\,{}_{m+2}C_3 + {}_{m+1}C_2$$

In a similar way we find that

$$1^4+2^4+\ldots+m^4 = 4!\,{}_{m+4}C_5 - 6\cdot 3!\,{}_{m+3}C_4 + 7\cdot 2!\,{}_{m+2}C_3 - {}_{m+1}C_2,$$
$$1^5+2^5+\ldots+m^5 = 5!\,{}_{m+5}C_6 - 10\cdot 4!\,{}_{m+4}C_5 + 25\cdot 3!\,{}_{m+3}C_4 - 15\cdot 2!\,{}_{m+2}C_3 + {}_{m+1}C_2,$$
$$1^6+2^6+\ldots+m^6 = 6!\,{}_{m+6}C_7 - 15\cdot 5!\,{}_{m+5}C_6 + 65\cdot 4!\,{}_{m+4}C_5 - 90\cdot 3!\,{}_{m+3}C_4 + 31\cdot 2!\,{}_{m+2}C_3 - {}_{m+1}C_2.$$

The above examples will give the reader a general idea of the power of the method discovered by Ben Ezra but used by him

in only a single case, and that case the very one in which it is least effective. This method was soon abandoned and, in the fourteenth century, we find Levi ben Gerson developing a theory of combinations very much like the one in present use.[1]

The results suggested by the fragment discussed above are as follows:

1. In the twelfth century there already existed a crude theory of combinations which owed its existence chiefly to the influence of mysticism.

2. The method used by Ben Ezra in finding the number of combinations of m elements taken n at a time was by reducing it to combinations of lower order according to a rule which may now be expressed as follows:

$$ {}_mC_n = {}_{m-1}C_{n-1} + {}_{m-2}C_{n-1} + {}_{m-3}C_{n-1} + \cdots + {}_{n-1}C_{n-1} $$

3. Ben Ezra was aware that the number of combinations of m elements taken 2 at a time is equivalent to the sum of the series

$$ 1 + 2 + \cdots + (m - 1), $$

and that

$$ {}_mC_3 = \frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \cdots + \frac{(m-2)\ (m-1)}{2} $$

but whether he knew the rule for higher cases is an open question.

4. Ben Ezra's work on permutations was one step in the history of the development of the theory of combinations, a process that was perfected in the time of Gersonides and his immediate successors.

[1] Attention to his work on permutations has been called by G. Enestrom, Bibl. Math., Vol. XIV (3), p. 276, but a thorough discussion of his achievement on this line is still lacking.